Worcester County Mathematics League

Varsity Meet 1 - October 27, 2021

Round 1 - Arithmetic

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Evaluate the expression below. Express your answer as an improper fraction $\frac{m}{n}$ in simplest terms.

$$\frac{1}{\frac{\frac{2}{3}}{4}} - \frac{\frac{1}{\frac{2}{3}}}{4}$$

2. Evaluate the expression below, expressing your answer as a decimal number.

$$\frac{1.8 \cdot 2.7 \cdot (0.6 + 3.6)}{(1.8 + 2.7) \cdot 0.6 \cdot 3.6}$$

3. Given the expression

$$2 - 3 \cdot 4 + 5$$

find all possible values of the expression that may result after placing one pair of parentheses, where the operations $-$, $\cdot$, and $+$ are binary operations. Thus, the placement $2(-3 \cdot 4) + 5$ is not allowed, while both $(2 - 3 \cdot 4 + 5)$ and $2 - (3 \cdot 4) + 5$ are possible placements.

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. {______________________}

Auburn, Quaboag, QSC

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. $a + b + c = 42$, and a, b, and c are in proportion $1 : 4 : 7$. Find $3a - 4b + c$.

2. A billboard measures 12 feet by $\dfrac{11}{3}$ feet. A border of uniform width is painted around the outer edges of the billboard. How wide is the border in feet if it covers one third of the area of the billboard?

3. How many different combinations of 100 bills of denominations \$100, \$10, and \$1 have a total value of \$1000?

ANSWERS

(1 pt) 1. $3a - 4b + c =$ ____________

(2 pts) 2. ____________________ ft.

(3 pts) 3. ____________________

Hopkinton, Clinton, Auburn

Worcester County Mathematics League

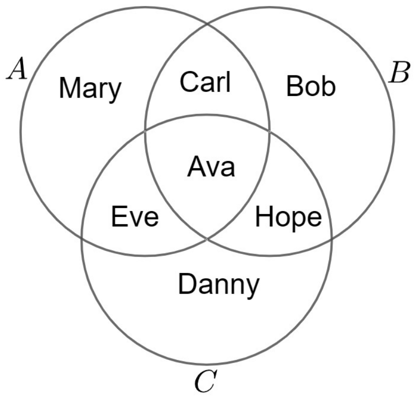
Varsity Meet 1 - October 27, 2021

Round 3 - Set Theory

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Given the Venn Diagram shown below for sets A, B, and C, find $(A \cup B) \cap (B \cup C)$.



2. Let the universal set U be the set of letters that appear in the words WORCESTER MATH LEAGUE, and let B be the set of four vowels (a, e, o, u). What is the number of subsets of B^C (the complement of B)?

3. There are 830 composite numbers less than 1000. Let S be the set of composite numbers that are not divisible by 2, 3, or 7. How many elements does S have?

ANSWERS

(1 pt) 1. {____________________}

(2 pts) 2. ____________________

(3 pts) 3. ____________________

QSC, Holliston, QSC

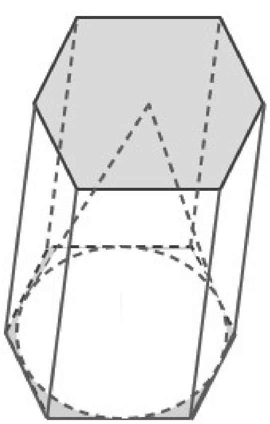
Worcester County Mathematics League
Varsity Meet 1 - October 27, 2021
Round 4 - Measurement

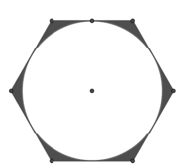
All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If the minute hand on a clock is 6 inches long and the tip of the minute hand moves a distance of $x\pi$ inches in 35 minutes, find x.

2. The bases of a trapezoid are 9 cm and 15 cm and the altitude is 7 cm. Find the area of the quadrilateral formed by joining the midpoints of the adjacent sides of the trapezoid.

3. Find the volume of the solid object shown below: a right hexagonal prism of height 10cm with a right cone of the same height cut from it. The base of the object is a regular hexagon with the inscribed circular base of the cone cut out, as shown to the right of the solid figure. The radius of the cone's base is 6cm. The volume of the figure can be written in simplest form as $a\sqrt{b} + c\pi cm^3$. Find the ordered triple (a, b, c).





ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________ cm^2

(3 pts) 3. (______________________)

Bromfield, Doherty, Notre Dame

Worcester County Mathematics League
Varsity Meet 1 - October 27, 2021
Round 5 - Polynomial Equations

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Polynomial $p(x) = x^2 + bx + c$ has integer coefficients b and c. If the roots of $p(x) = 0$ are $3 + \sqrt{2}$ and $3 - \sqrt{2}$, find the ordered pair (b, c).

2. Find all integer solutions to the following equation:

$$\left|x + 3 + 2\sqrt{5}\right| \cdot \left|x + 3 - 2\sqrt{5}\right| = 16$$

3. Let the roots of $x^2 + kx + 12 = 0$ be r_1 and r_2 and let the roots of $x^2 - kx + 12 = 0$ be r_3 and r_4. Find all possible values of k if

$$f(r_1, r_2, r_3, r_4) = r_1^2 + r_2^2 + r_3^2 + r_4^2 - 2r_1r_3 - 2r_1r_4 - 2r_2r_3 - 2r_2r_4 + 8(r_1 + r_2 - r_3 - r_4) = 0$$

ANSWERS

(1 pt) 1. (____________________)

(2 pts) 2. { ____________________ }

(3 pts) 3. $k \in$ { ____________________ }

Tantasqua, St. Johns, QSC

Worcester County Mathematics League
Varsity Meet 1 - October 27, 2021
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Evaluate
$$84 \div (12 + 6 - 3 \cdot 3 - 7 + 4)$$

2. Solve the following system for x and y and express your answer as the ordered pair (x, y).
$$\frac{1}{x} + \frac{1}{y} = 5$$
$$\frac{2}{x} + \frac{3}{y} = 13$$

3. Babel High School requires each of its 650 students take a foreign language and no student is allowed to take more than two. It offers four foreign languages. If 125 students take German, 185 take Chinese, and 245 take Spanish and 90 students take exactly 2 languages, how many students take French, the fourth language?

4. A regular octagon has sides of length 4cm. The area of the octagon can be expressed in simplest terms as $a + b\sqrt{c}$, where a and b are rational numbers and c is a positive integer. Find $a + b + c$.

5. Find all real solutions of the following equation:
$$3x - 2\sqrt{3x} - 8 = 0$$

6. Let m and n be integers and x and y be three digit numbers such that $x = 2^m$ and $y = 5^n$ such that the second digit of x is the same as the first digit of y. Find the ordered pair (x, y).

7. The sum of two numbers is 12. Their product is 32. What is the sum of their reciprocals?

8. A spherical ball of radius 10cm is placed on an open box. The dimensions of the open side of the box are 10cm X 10cm and the height of the box is 8cm. The distance from the ball to the bottom of the box can be expressed in the form $a + b\sqrt{c}$. Find the ordered triple (a, b, c).

9. If $x - 4$ divides $p(x) = x^3 - kx + 8$ with a remainder of zero, find k.

Assabet Valley, Shepherd Hill, Algonquin, Shrewsbury, Tantasqua, Burncoat, Bromfield, QSC, Westborough

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. ________________________

2. (______________________)

3. ________________________

4. $a + b + c =$ ______________

5. ________________________

6. (______________________)

7. {______________________}

8. (______________________)

9. ________________________

Worcester County Mathematics League
Varsity Meet 1 - October 27, 2021
Answer Key

Round 1 - Arithmetic

1. $\frac{45}{8}$
2. 2.1
3. $\{-25, -15, -5, 1\}$ (any order)

Round 2 - Algebra I

1. -21
2. $\frac{1}{2}$
3. 10

Round 3 - Set Theory

1. {Ava, Bob, Carl, Eve, Hope}
2. 512
3. 120

Round 4 - Measurement

1. 7
2. 42
3. $(720, 3, -120)$ (order matters)

Round 5 - Polynomial Equations

1. $(-6, 5)$ (order matters)
2. $\{-9, -5, -1, 3\}$ (any order, need all four)
3. $\{-2, 6\}$ (any order, need both)

Team Round

1. 14
2. $\left(\frac{1}{2}, \frac{1}{3}\right)$ (in this order)
3. 185
4. 66
5. $\frac{16}{3}$ (only real solution)
6. $(512, 125)$ (order matters)
7. $\{4, 8\}$ (either order)
8. $(-2, 5, 3)$ (order matters)
9. 18

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Solutions

Round 1 - Arithmetic

1. Apply the reciprocal rule $\left(\frac{1}{\frac{b}{a}} = \frac{a}{b}\right)$ multiple times to simplify the expression:

$$\frac{1}{\frac{\frac{2}{3}}{4}} - \frac{\frac{1}{\frac{2}{3}}}{4} = \frac{4}{\frac{2}{3}} - \frac{\frac{3}{2}}{4} = \frac{3}{2}(4) - \frac{3}{2\cdot 4} = 6 - \frac{3}{8}$$

$$= \frac{48}{8} - \frac{3}{8} = \boxed{\frac{45}{8}}$$

2. $$\frac{1.8\cdot 2.7\cdot(0.6+3.6)}{(1.8+2.7)\cdot 0.6\cdot 3.6} = \frac{(\cancel{1.8}^{\,1})(2.7)(4.2)}{(4.5)(0.6)(\cancel{3.6}_{\,2})} = \frac{2.7(\cancel{4.2}^{\,7})}{(4.5)(\cancel{0.6}_{\,1})2}$$

$$= \frac{2.7(7)}{(4.5)2} = \frac{(2.7)\cdot 7}{9} = 0.3\cdot 7 = \boxed{2.1}$$

The arithmetic is simpler if terms are cancelled from the numerator and denominator before multiplying the products, as demonstrated above.

3. There are 3 binary operations in the expression and one pair of parentheses can be placed six different ways: one around all three operations, twice around two operations, and three ways around a single operation:

$(2-3\cdot 4+5) = (2-12+5) = -5$

$(2-3\cdot 4)+5 = (2-12)+5 = -5$

$2-(3\cdot 4+5) = 2-(12+5) = -15$

$(2-3)\cdot 4+5 = (-1)\cdot 4+5 = -4+5 = 1$

$2-(3\cdot 4)+5 = 2-12+5 = -5$

$2-3\cdot(4+5) = 2-3(9) = 2-27 = -25$

Removing the repetitions of -5, there are four values: $\boxed{\{-25,-15,-5,1\}}$

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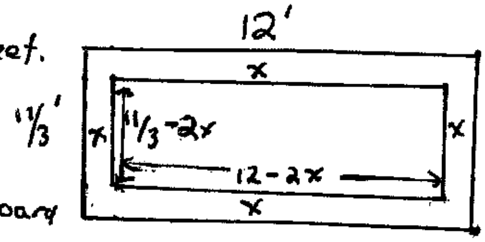
Round 2 Algebra I Solutions

1. Let the scale factor for $a:b:c$ be x, so that $a=x$, $b=4x$ and $c=7x$. Then $a+b+c=x+4x+7x=12x=42$ and $x=\frac{42}{12}=\frac{7}{2}$. Then $3a-4b+c=3x-4(4x)+7x=10x-16x=-6x=-6(7/2)=\boxed{-21}$

2. The billboard is shown at the right with a uniform border of width x feet. Note that the area of the billboard is $12(11/3)=44\ ft^2$. It is easier to calculate the area of the billboard that is inside the border, rather than the border area.



This area is $(\frac{11}{3}-2x)(12-2x)$ and is equal to two thirds of the size of the billboard, or $\frac{2}{3}(44)=\frac{88}{3}\ ft^2$. Setting these equal:

$$\left(\frac{11}{3}-2x\right)(12-2x)=4x^2-24x-\frac{22}{3}x+44=\frac{88}{3}$$

Multiply this equation by $3/2$ to convert to integer coefficients:

$$\frac{3}{2}\left(4x^2-24x-\frac{22}{3}x+44=\frac{88}{3}\right)=6x^2-36x-11x+66=44$$

or,

$$6x^2-47x+22=0$$

This quadratic can be factored by grouping since $6\cdot 22=3\cdot 44$ and $3+44=47$, so

$$6x^2-3x-44x+22=3x(2x-1)-22(2x-1)=(3x-22)(2x-1)=0$$

The two roots are $\frac{22}{3}$ and $\frac{1}{2}$. $x=\frac{22}{3}$ results in negative dimensions and is extraneous. Thus, $\boxed{x=1/2}$

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Round 2 Algebra I Solutions (cont.)

3. Let the number of \$1, \$10, and \$100 bills be a, b, and c, respectively. Then:

$$a + 10b + 100c = 1000$$
$$a + b + c = 100$$

Subtracting the second equation from the first gives:

$$9b + 99c = 900; \quad \text{or } b + 11c = 100$$

Now c must be less than 10 since b cannot be negative. Once the value of c is selected, $b = 100 - 11c$, and $a = 100 - b - c$.

Thus, there are $\boxed{10}$ possible combinations of 100 bills of denominations \$1, \$10, and \$100 that have a total value of \$1000

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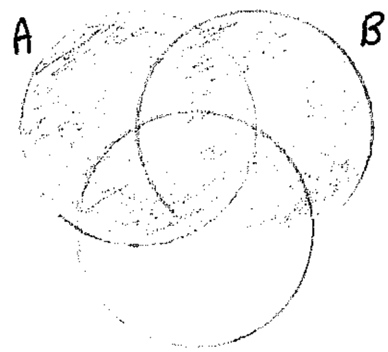
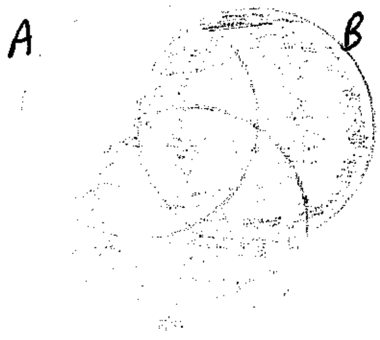
Round 3 Set Theory Solutions

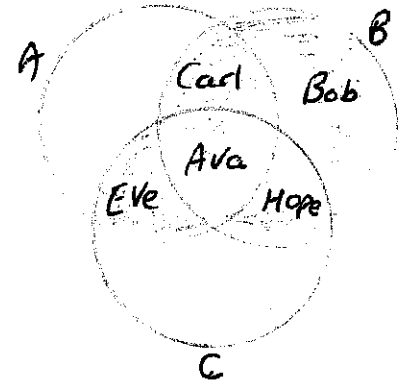
1



$A \cup B$



$B \cup C$



$(A \cup B) \cap (B \cup C)$

The set $(A \cup B) \cap (B \cup C)$ is found graphically above. The two unions $A \cup B$ and $B \cup C$ are shaded in the left and center Venn diagrams. The desired set is shaded in the right diagram; the shaded area is the area common to the two union diagrams because it is their intersection. Thus, $(A \cup B) \cap (B \cup C) = \boxed{\{\text{Ava, Bob, Carl, Eve, Hope}\}}$

2. $U = \{W, O, R, C, E, S, T, M, A, H, L, G, U\}$

B^c is the set of elements in U and <u>not</u> in B: $B^c = \{W, R, C, S, T, M, H, L, G\}$

There are 9 elements in B^c. The number of subsets of $B^c = 2^9$, or $\boxed{512}$.

3. Let S_1 = set of composite numbers divisible by 2 and less than 1000

S_2 = set of composite numbers divisible by 3 and less than 1000

S_3 = set of composite numbers divisible by 7 and less than 1000

U = set of composite numbers less than 1000, where $|U| = 830$

Now S^c = Complement of $S = S_1 \cup S_2 \cup S_3$. We will find the count of S^c and subtract that number from $|U| = 830$. Applying the principle of Inclusion–exclusion:

$$|S^c| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_1 \cap S_3| + |S_1 \cap S_2 \cap S_3|$$

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Round 3 Set Theory Solutions (cont.)

Now: $|S_1| = \left\lfloor \frac{999}{2} \right\rfloor - 1$ where $\lfloor \cdot \rfloor$ indicates "round down" to the nearest integer, and 1 is subtracted because 2 is prime and not composite.

$= 499 - 1 = 498$

$|S_2| = \left\lfloor \frac{999}{3} \right\rfloor - 1 = 333 - 1 = 332$

$|S_3| = \left\lfloor \frac{999}{7} \right\rfloor - 1 = 142 - 1 = 141$

$|S_1 \cap S_2| = \left\lfloor \frac{999}{6} \right\rfloor = 166$ because numbers divisible by both 2 and 3 are divisible by 6 and composite.

$|S_2 \cap S_3| = \left\lfloor \frac{999}{21} \right\rfloor = 47$

$|S_1 \cap S_3| = \left\lfloor \frac{999}{14} \right\rfloor = 71$

$|S_1 \cap S_2 \cap S_3| = \left\lfloor \frac{999}{42} \right\rfloor = 23$

So $|S^c| = \underbrace{498 + 332 + 141}_{971} - \underbrace{166 - 47 - 71}_{284} + 23 = 994 - 284 = 710$

$|S| = |U| - |S^c| = 830 - 710 = \boxed{120}$

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Solutions

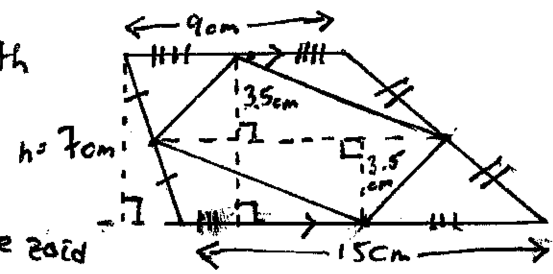
Round 4 (Measurement)

1. The tip of the minute hand traces an arc of a circle of radius 6 inches. The length of the arc is $\frac{35}{60}$ of the circumference of the circle, C, because it takes 60 minutes for the minute hand tip to trace the full circumference of the circle. For a circle, $C = 2\pi r$, where r is the radius of the circle. Thus:

$$x\pi = \left(\frac{35}{60}\right)2\pi 6 = \frac{35(2\pi)}{10} = \frac{70\pi}{10} = 7\pi$$

And $x = \boxed{7}$

2. A trapezoid is drawn at right with the inscribed quadrilateral whose vertices are midpoints of the four sides. The median of the trapezoid is parallel to the two bases and its length m is half the sum of the base lengths: $m = \frac{1}{2}(9+15) = 12\text{cm}$



The median splits the quadrilateral into two triangles with a shared base of length m. Each triangle's altitude to the shared base is half the height of the trapezoid because the median divides all transversals of the two parallel bases into congruent segments. Adding the areas of the triangles: $\frac{1}{2}m(h/2) + \frac{1}{2}mh/2 = (mh)/2 = \frac{12\cdot 7}{2} = \boxed{42\text{cm}^2}$

Note that we have shown that the area of the quadrilateral is exactly one half the area of the trapezoid.

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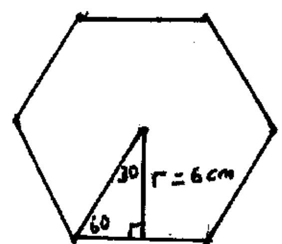
Round 4 Measurement solutions (cont.)

3\. Let V_p be the volume of the prism and V_c be the volume of the cone.

Then the desired volume is $V_p - V_c$.

Let B = area of the hexagonal base of the prism.
and h = height of the prism = height of the cone = 10cm
r = radius of the circular base of the cone = 6cm

The volume of a prism $V_p = Bh$.
The volume of a cone $V_c = \frac{1}{3}\pi r^2 h$



The area of the hexagonal prism base $B = \frac{1}{2}pa$

where $p = 6s$ = the perimeter of the hexagon
S = length of one side of the regular hexagon
a = apothem = r, the radius of the inscribed circle.

The side length is found using the 30-60-90 triangle shown in the diagram:

$\frac{S}{2} = \frac{r}{\sqrt{3}} = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3}\sqrt{3}} = 2\sqrt{3}$, so $S = 4\sqrt{3}$ and $p = 24\sqrt{3}$. Then

$$V_p = \left(\tfrac{1}{2}pa\right)h = \tfrac{1}{2}(24\sqrt{3})(6)10 = 720\sqrt{3} \text{ cm}^3$$

$$V_c = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi 6^2 10 = 120\pi \text{ cm}^3$$

$$V = V_p - V_c = 720\sqrt{3} - 120\pi \text{ cm}^3 = a\sqrt{b} + c\pi$$

So $(a, b, c) = \boxed{(720, 3, -120)}$ (order matters)

Note: B may also be found by noting that the diagonals of a regular hexagon split the hexagon into six congruent equilateral triangles of height 6cm and side length $4\sqrt{3}$ cm, so

$$B = 6\left(\tfrac{1}{2}bh\right) = 3(4\sqrt{3})6 = 72\sqrt{3}.$$

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Round 5 Polynomial Equations Solutions

1. $p(x) = (x - r_1)(x - r_2)$ in factored form, where r_1 and r_2 are the two roots of $p(x) = 0$. Inserting the given roots and distributing:

$$\begin{aligned}(x - (3+\sqrt{2}))(x - (3 - \sqrt{2})) &= x^2 - x(3+\sqrt{2}) - x(3-\sqrt{2}) + (3+\sqrt{2})(3-\sqrt{2}) \\ &= x^2 - 3x - 3x - x\sqrt{2} + x\sqrt{2} + 3^2 - (\sqrt{2})^2 \\ &= x^2 - 6x + 9 - 4 \\ &= x^2 - 6x + 5\end{aligned}$$

and $(b, c) = \boxed{(-6, 5)}$ (order matters)

Alternatively, b is the negative sum of the roots and c is the product of the roots, as per Viete's formula.

2. Note that $|a| \cdot |b|$ has two possible values: $(+a)(+b) = (-a)(-b) = ab$ and $(+a)(-b) = (-a)(+b) = -ab$. Thus, the given absolute value equation is solved by finding all solutions to each of two equations:

$$(x + 3 + 2\sqrt{5})(x + 3 - 2\sqrt{5}) = 16$$
$$(x + 3 + 2\sqrt{5})(x + 3 - 2\sqrt{5}) = -16$$

Note that $(x + 3 + 2\sqrt{5})(x + 3 - 2\sqrt{5}) = x^2 + (3 + 2\sqrt{5} + 3 - 2\sqrt{5})x + (3 + 2\sqrt{5})(3 - 2\sqrt{5})$
$= x^2 + 6x + 3^2 - (2\sqrt{5})^2 = x^2 + 6x + 9 - 20 = x^2 + 6x - 11$

Solving each of the two equations separately:

$x^2 + 6x - 11 = 16$	$x^2 + 6x - 11 = -16$
$x^2 + 6x - 27 = 0$	$x^2 + 6x + 5 = 0$
$(x + 9)(x - 3) = 0$	$(x + 5)(x + 1) = 0$
$x = -9$ or 3	$x = -5$ or -1

and $x \in \boxed{\{-9, -5, -1, 3\}}$ need all four, any order.

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Round 5 Polygnomial Equations Solutions (cont.)

3. First, note that $r_1 + r_2 - r_3 - r_4 = (r_1 + r_2) - (r_3 + r_4) = -k - k = -2k$ because the sum of the roots of $x^2 + kx + 12 = 0$ is $-k$ and the sum of the roots of $x^2 - kx + 12 = 0$ is k, due to Viete's formula.

Also, $r_1 r_2 = r_3 r_4 = 12$ from Viete's formula, which will be used later

The solutions are found by expressing $f(r_1, r_2, r_3, r_4)$ as a function of $r_1 + r_2 - r_3 - r_4 = -2k$, and noting that

$$(r_1 + r_2 - r_3 - r_4)^2 = ((r_1 + r_2) - (r_3 + r_4))^2$$
$$= (r_1 + r_2)^2 - 2(r_1 + r_2)(r_3 + r_4) + (r_3 + r_4)^2$$
$$= r_1^2 + 2r_1 r_2 + r_2^2 - 2r_1 r_3 - 2r_1 r_4 - 2r_2 r_3 - 2r_2 r_4 + r_3^2 + 2r_3 r_4 + r_4^2$$
$$= r_1^2 + r_2^2 + r_3^2 + r_4^2 - 2r_1 r_3 - 2r_1 r_4 - 2r_2 r_3 - 2r_2 r_4 + 2r_1 r_2 + 2r_3 r_4$$

Note that $f(r_1, r_2, r_3, r_4)$ is equal to the expression for $(r_1 + r_2 - r_3 - r_4)^2$ plus $8(r_1 + r_2 - r_3 - r_4) - 2r_1 r_2 - 2r_3 r_4$. Also, recall that $-2r_1 r_2 - 2r_3 r_4 = -2(12) - 2(12) = -48$. Thus:

$$f(r_1, r_2, r_3, r_4) = (r_1 + r_2 - r_3 - r_4)^2 + 8(r_1 + r_2 - r_3 - r_4) - 2r_1 r_2 - 2r_3 r_4$$
$$= (-2k)^2 + 8(-2k) - 2(12) - 2(12)$$
$$= 4k^2 - 16k - 48 = 0$$

Simplifying the polynomial and factoring:

$$4k^2 - 16k - 48 = 0$$
$$k^2 - 4k - 12 = 0$$
$$(k-6)(k+2) = 0$$

And $\boxed{k \in \{-2, 6\}}$

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Team Round Solutions

1. $84 \div (12+6-3\cdot 3-7+4) = 84 \div (\underbrace{12+6}-\underbrace{9-7}+4) = 84 \div (18-16+4) = 84 \div (2+4)$
$= 84 \div 6 = \boxed{14}$

2. Let $u = \frac{1}{x}$ and $V = \frac{1}{y}$, solve for u and v, then x and y:

$$\left.\begin{aligned} u+V&=5 \\ 2u+3V&=13 \end{aligned}\right\} \quad \begin{aligned} &2u+3V=13 \\ -&2(u+V=5) \\ \hline \end{aligned} \rightarrow \begin{aligned} &2u+3V=13 \\ -&(2u+2V=10) \\ \hline &V=3 \end{aligned} \qquad u=5-v=5-3=2$$

$x = \frac{1}{u} = \frac{1}{2}$; $y = \frac{1}{v} = \frac{1}{3} \Rightarrow (x,y) = \boxed{(1/2, 1/3)}$

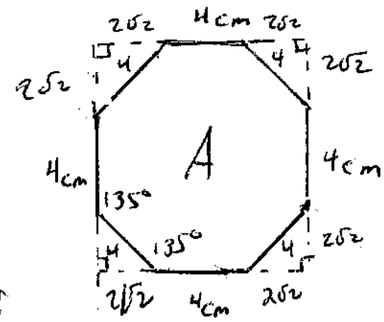
3. By the Principal of Inclusion-Exclusion:

$$650 = \text{Total Number of Students} = \underbrace{125}_{\text{take German}} + \underbrace{185}_{\text{take Chinese}} + \underbrace{245}_{\text{take Spanish}} + x - \underbrace{90}_{\text{Take two languages}} + \underbrace{0}_{\text{Take three languages}} - \underbrace{0}_{\text{Take four}}$$

x = # students who take French

$= 650 - 125 - 185 - 245 + 90 = 650 - 310 - 155 = 340 - 155 = \boxed{185}$

4. The octagon area can be found by extending four of the side until they intersect and form a square. The area of the octagon is the area of the square minus the areas of the four "corner" triangles.



Note that the corner triangles are 45-45-90 triangles because the interior angles of a regular octagon measure $\frac{(8-2)\cdot 180}{8} = \frac{1080}{8} = 135°$ and the exterior angles measure $180° - 135° = 45°$.

One side of the square is $4+4\sqrt{2}$ because the corner triangle sides are $2\sqrt{2}$, $2\sqrt{2}$, and 4, in proportion $1:1:\sqrt{2}$. The corner triangles have area $\frac{1}{2}(2\sqrt{2})^2 = \frac{8}{2} = 4$. The square area $= (4+4\sqrt{2})^2 = 16+32+2(16\sqrt{2})$
$= 48 + 32\sqrt{2}$

A = octagon area $= A_{square} - 4A_{corner} = 48 + 32\sqrt{2} - 4\cdot 4 = 48 - 16 + 32\sqrt{2}$
$= \boxed{32+32\sqrt{2}}$ cm2 and $a+b+c = 32+32+2 = \boxed{66}$

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Team Round Solutions (cont.)

5. Let $z = \sqrt{3x}$. Then $z^2 = 3x$. Substituting in the given equation:

$$z^2 - 2z - 8 = 0$$

Factoring: $(z-4)(z+2)=0$ and $z=4$ or $z=-2$. The second solution is extraneous because $z = \sqrt{3x} \geq 0$. So $z = 4 = \sqrt{3x}$, and $16 = 3x$

So $\boxed{x = 16/3}$ only real solution

6. $x = 2^m$. for $m \leq 6$, $2^m \leq 2^6 = 64$; for $m \geq 10$, $2^m \geq 1024$ Thus $m \in \{7, 8, 9\}$

$x = 2^m \in \{2^7, 2^8, 2^9\} = \{128, 256, 512\}$

Likewise $y = 5^n$ for $n < 3$, $5^n < 100$; for $n > 4$, $5^n > 1000$

$y = 5^n \in \{5^3, 5^4\} = \{125, 625\}$ Note that the first digit of y is equal to 1 or 6; 1 is the second digit of 512, but no power of two with three digits has a second digit equal to 6. Therefore

$x = 512$, $y = 125$, or $(x, y) = \boxed{(512, 125)}$

7. Let the two numbers be x and y. Then $x + y = 12$ and $y = 12 - x$. Substituting for y in the product equation results in a quadratic:

$$xy = 32, \quad x(12-x) = 32, \quad 12x - x^2 = 32$$

Rearrange and factor the quadratic equation:

$$x^2 - 12x + 32 = 0, \quad (x-4)(x-8) = 0$$

And $x = 4$ or 8 (y is the other one. The answer is $\boxed{\{4, 8\}}$ either order.

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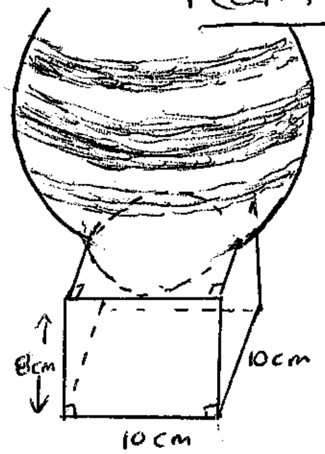
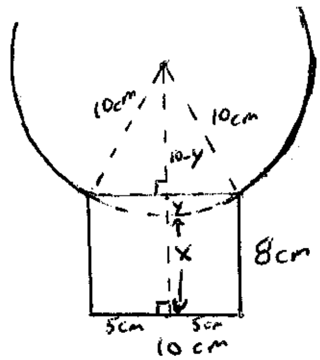
Team Round Solutions (cont.)

8.



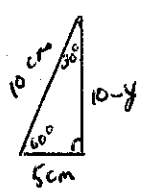
ball on box



cross section

The ball sits symmetrically in the box opening, as drawn on the left. The plane of the open box top intersects the ball in a circle that is inscribed in the box top square.

A cross section of the box and sphere is drawn on the right, where the cross section is created by the plane parallel to and halfway between the front and back of the box. Radii of 10cm are drawn from the center of sphere to the edges of the box opening, where the cross sections of the sphere and the box are a circle of radius 10cm and a 10cm X 8cm rectangle. A dashed line is drawn from sphere center to the center of the box bottom. The desired distance from the ball to the box bottom is labeled x, and y is the distance that the ball extends below the plane of the box top. Then $x = 8\text{cm} - y$, and y is found from the right triangle shown at the left, which happens to be a 30-60-90 triangle. Then



$$10 - y = 5\sqrt{3}, \quad y = 10 - 5\sqrt{3}, \quad x = 8 - y = 8 - (10 - 5\sqrt{3}) = 5\sqrt{3} - 2$$

or $x = -2 + 5\sqrt{3}$, so $(a, b, c) = \boxed{(-2, 5, 3)}$

9. Perform synthetic division on $\dfrac{x^3 - kx + 8}{x - 4}$ $\longrightarrow$ set the remainder to 0:

$$\begin{array}{r|rrrr} 4 & 1 & 0 & -k & 8 \\ & & 4 & 16 & 64-4k \\ \hline & 1 & 4 & 16-k & 72-4k \end{array}$$

$72 - 4k = 0$

$4k = 72$

$\boxed{k = 18}$